The Statement of the Prime Number Theorem

Chebyshev's result $\pi(x) > ax/\log x$ is far stronger than $\pi(x) > c\log x$, seen in Problem Sheet 1, yet is still a long way from the truth. It will be shown that

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{i.e.} \quad \lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

This is the ***Prime Number Theorem***, conjectured by Euler in 1762, Gauss in 1791 and Legendre in 1798 and proved independently by J. Hadamard and C. de la Vallée-Poussin in 1896.

We will not in fact prove the Prime Number Theorem for $\pi(x)$ but, just as we deduced bounds on $\pi(x)$ from those on $\psi(x)$, we will prove in the following sections that

$$\psi(x) \sim x.$$

This is equivalent to the Prime Number Theorem as we will now show.

Corollary 2.25

$$\pi(x) \sim \frac{x}{\log x} \quad \textit{if, and only if,} \quad \psi(x) \sim x.$$

Proof From (13) and (15) we get

$$\pi(x) = \frac{\psi(x) + O\left(x^{1/2}\right)}{\log x} + O\left(\frac{x}{\log^2 x}\right) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Thus

$$\frac{\pi(x)}{x/\log x} = \frac{\psi(x)}{x} + O\left(\frac{1}{\log x}\right).$$

Therefore, on the assumption that the limits exist, they must satisfy

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = \lim_{x\to\infty} \frac{\psi(x)}{x}.$$

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Aside on Prime Number Theorem

Though $x/\log x$ is a good approximation to $\pi(x)$ it is not a *very good* approximation. For a better approximation recall

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \theta(t)\,\frac{dt}{t\log^2 t}. \tag{24}$$

The prime number theorem in the form $\psi(x) \sim x$ is, by Lemma 2.16, equivalent to $\theta(x) \sim x$, which 'suggests' replacing $\theta(x)$ in (24) by x. This would 'suggest' an approximation to $\pi(x)$ of

$$\frac{x}{\log x} + \int_2^x t\frac{dt}{t\log^2 t} \;=\; \frac{x}{\log x} + \left[t\left(-\frac{1}{\log t}\right)\right]_2^x + \int_2^x \frac{dt}{\log^2 t} \;=\; \int_2^x \frac{dt}{\log t} + \frac{2}{\log 2},$$

having integrated by parts, starting by integrating $1/\left(t\log^2 t\right)$. The better approximation to $\pi(x)$ may thus be given by the **logarithmic integral**

$$\mathrm{li}x = \int_2^x \frac{dt}{\log t}.$$

Looking back at its derivation above,

$$\mathrm{li}x = \frac{x}{\log x} - \frac{2}{\log s} + \int_2^x \frac{dt}{\log t}.$$

The integral here can be estimated by splitting at $\sqrt{x}$, as seen in the proof of Theorem 2.20, giving

$$\mathrm{li}x = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

By repeated integration by parts and an estimation of the final integral by splitting at $\sqrt{x}$, we have

$$\mathrm{li}x = \int_2^x \frac{dt}{\log t} = \sum_{j=0}^{m} j!\frac{x}{\log^{1+j} x} + O_m\left(\frac{x}{\log^{m+2}}\right).$$

for $m \geq 1$. **Be careful,** in some books $\mathrm{li}x$ denotes the integral

$$\int_0^x \frac{dt}{\log t} = \lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right),$$

whilst other books call this latter integral $\mathrm{Li}x$.